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An eigenvalue problem in two dimensions for an irregular boundary

S Chakraborty¹, J K Bhattacharjee² and S P Khastgir³

¹ Saha Institute of Nuclear Physics, 1/AF, BidhanNagar, Kolkata 700 064, India

² SN Bose National Centre for Basic Sciences, Sector-III, Block-JD, Salt Lake, Kolkata 700 098, India

³ Department of Physics and Meteorology, Indian Institute of Technology, Kharagpur 721 302, India

E-mail: somdeb.chakraborty@saha.ac.in, jayanta.bhattacharjee@gmail.com and pratik@phy.iitkgp.ernet.in

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Abstract

An analytical perturbative method is suggested for solving the Helmholtz equation $(\nabla^2 + k^2)\psi = 0$ in two dimensions where ψ vanishes on an irregular closed curve. We can thus find the energy levels of a quantum mechanical particle confined in an infinitely deep potential well in two dimensions having an irregular boundary or the vibration frequencies of a membrane whose edge is an irregular closed curve. The method is tested by calculating the energy levels for an elliptical and a supercircular boundary and comparing with the results obtained numerically. Further, the phenomenon of level crossing due to shape variation is also discussed.

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1. Introduction

The energy levels of a quantum particle confined in a 2D regular box can be solved exactly only in the cases of a square and a triangle and in the limiting case of a circle. While the determination of the energy levels for the circular or the square boundary is a trivial exercise, the problem of the triangular boundary is more formidable [1]. The corresponding problems in the classical regime can be the flow of liquid through a pipe of a polygonal cross-section or the free vibration of a membrane (with a fixed boundary) of a polygonal shape. The classical problems, like their quantum counterparts, are amenable to simple analytical treatments only in the cases of a circle, a square and a triangle. The problem of a regular polygonal box has been solved by perturbing about the equivalent circle and the results have been quite accurate [2]. The same problem has been solved by Cureton and Kuttler [3] in the context of vibration

of membranes. Here we address the problem of finding out the energy eigenvalues when the boundary has no simple geometric shape. The Schrödinger equation for a particle of mass m and energy E confined in an infinitely deep 2D potential well is

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E\psi. \quad (1a)$$

The above equation can be recast as

$$(\nabla^2 + k^2)\psi = 0, \quad (1b)$$

where $k = \sqrt{\frac{2mE}{\hbar^2}}$. Thus the problem boils down to solving the Helmholtz equation with the Dirichlet condition $\psi = 0$ on the ‘irregular’ boundary. Exact solutions can be obtained only in a few special cases as mentioned earlier. The standard procedure is to choose a curvilinear coordinate system suited to the geometry of the problem and employ the method of separation of variables. For a boundary having an irregular shape no particular coordinate system will be useful. Hence, we resort to perturbative methods to solve the problem. Here we will perturb the boundary about a circle so that in our problem solutions can be obtained in the form of corrections to the solutions for the circular boundary. Until now, most of the efforts at finding out the eigenvalues of the Helmholtz equation for an irregular boundary have been numerical. Mazumdar [4–6] reviews the approximate methods invoked for this problem. In addition to an extensive summary of theoretical results, Kuttler and Sigillito [7] also give a comprehensive review of the different numerical methods employed. More recently, Amore [8] gives a numerical recipe using a collocation approach based on little sinc functions. As far as analytical works are concerned, Rayleigh [9] and also Fetter and Walecka [10] find the ground state energy eigenvalues for a vibrating membrane. A general formalism has been suggested by Morse and Feshbach [11] using the Green functions. Parker and Mote [12] have put forward a perturbative method for finding the eigenvalues and the eigenfunctions through fifth order. A similar method has been proposed by Nayfeh [13]. However, the eigenvalues are found out only to the first order. Read [14] has also suggested a general analytical approach to the problem. Bera *et al* [15] have proposed a perturbative approach to the problem but failed to express the solutions in a closed form. Our approach is similar in spirit to that of Bera. Here we present a solution to the problem in a more systematic and efficient manner. The perturbative correction to the eigenvalues and the eigenfunctions is presented in a closed form at each order of perturbation. The method is tested by comparing the analytical results with those obtained numerically for a supercircular and an elliptical boundary. Further, the phenomenon of energy level crossing as induced by the shape variation is also dealt with for both the boundaries. In section 2 we set up our general scheme and in section 3 we apply it to the cases of a supercircle and an ellipse. A short conclusion is presented in section 4.

2. Perturbation about the equivalent circle

It was shown by Rayleigh [9] that the fundamental frequency of a membrane whose boundary is not extravagantly elongated is nearly same as that of a mechanically similar circular membrane having the same area. The above result naturally leads us to develop, in the following, a perturbation about the equal area circle.

Given, any $r(\theta) = r(\theta + 2\pi)$, defining the boundary in 2D enclosing an area, $A = \frac{1}{2} \int_0^{2\pi} r^2(\theta) d\theta$, we first construct a circle of radius, R_0 , such that

$$A = \pi R_0^2. \quad (2)$$

We can then expand $r(\theta)$ about R_0 in terms of Fourier series at different orders of smallness (denoted by λ) as

$$r(\theta) = R_0 \left[1 + \sum_{\sigma=1}^{\infty} \lambda^{\sigma} f^{(\sigma)}(\theta) \right], \tag{3}$$

where

$$f^{(\sigma)}(\theta) = \sum_{n=0}^{\infty} (C_n^{(\sigma)} \cos n\theta + S_n^{(\sigma)} \sin n\theta). \tag{4}$$

Here, for simplicity, we have considered a one parameter (deformation parameter), λ , dependence of $r(\theta)$, which thus represents a family of curves which reduce to the equation for a circle in the limiting case $\lambda \rightarrow 0$. In principle, λ should be much smaller than unity ensuring that the variation of $r(\theta)$ with θ is small enough to permit the use of perturbative methods. However, as we will see in the following section that for the case of a supercircle $\lambda \sim 1$ works quite well and keeps the results within 10% error. We also note here that the Fourier expansion of the boundary in (3) is rather unusual and makes our method different from all other existing methods. Here in fact each $f^{(\sigma)}(\theta)$ is a Fourier series in itself of order λ^{σ} . Earlier methods in the literature had worked with only one Fourier series—that is by summing all the orders into one. The main advantage in treating the problem like this is to have an analogy with the time-independent perturbation scheme of quantum mechanics and obtain closed form solutions at each order of λ . If we now calculate the area using (3), (4) and equate it with πR_0^2 , we arrive at the following constraint relations among the Fourier coefficients:

$$\sum_{n=0}^{\infty} \sum_{v=1}^{\sigma-1} [C_n^{(v)} C_n^{(\sigma-v)} + 2C_0^{(v)} C_0^{(\sigma-v)} + S_n^{(v)} S_n^{(\sigma-v)}] = -4C_0^{(\sigma)}. \tag{5a}$$

In particular we have,

$$C_0^{(1)} = 0, \tag{5b}$$

and

$$4C_0^{(2)} = - \sum_{n=1}^{\infty} [C_n^{(1)2} + S_n^{(1)2}]. \tag{5c}$$

Now, as a first approximation, the energy E_0 of the particle confined by $r(\theta)$ will be that of a particle enclosed in a circle of radius R_0

$$E_0 = \frac{\hbar^2 \rho_{l,j}^2}{2m R_0^2}, \tag{6}$$

with $\rho_{l,j} = k_{l,j} R_0$ being the j th node of the l th order Bessel function. The next step is to improve upon the ‘equal area’ approximation by perturbing the equivalent circle and finding out the first- and the second-order corrections to the eigenvalues.

We now treat λ as the perturbation parameter and expand ψ and E as

$$\psi = \psi_0 + \lambda \psi_1 + \lambda^2 \psi_2 + \dots, \tag{7a}$$

$$E = E_0 + \lambda E_1 + \lambda^2 E_2 + \dots. \tag{7b}$$

Using (7a), (7b) in (1a), equating the coefficients of different powers of λ to 0 and after some rearrangement we arrive at the set of equations

$$\left(\nabla^2 + \frac{2mE_0}{\hbar^2}\right)\psi_0 = 0, \tag{8a}$$

$$\left(\nabla^2 + \frac{2mE_0}{\hbar^2}\right)\psi_1 = -\frac{2mE_1}{\hbar^2}\psi_0, \tag{8b}$$

$$\left(\nabla^2 + \frac{2mE_0}{\hbar^2}\right)\psi_2 = -\frac{2m}{\hbar^2}(E_1\psi_1 + E_2\psi_0). \tag{8c}$$

Equation (8a) can readily be identified as the equation for the circular boundary with ψ_0 as the eigenfunction corresponding to energy E_0 .

The boundary condition is

$$\psi(R_0 + \lambda R_0 f^{(1)} + \lambda^2 R_0 f^{(2)} + \dots) = 0.$$

Taylor expanding about $r = R_0$, with (7a) and equating the coefficients of different powers of λ to 0, we find

$$\psi_0(R_0) = 0, \tag{9a}$$

$$\psi_1(R_0) + R_0 f^{(1)} \psi_0'(R_0) = 0, \tag{9b}$$

$$\psi_2(R_0) + R_0 f^{(1)} \psi_1'(R_0) + R_0 f^{(2)} \psi_0'(R_0) + \frac{1}{2} R_0^2 f^{(1)2} \psi_0''(R_0) = 0. \tag{9c}$$

We discuss separately the cases $l = 0$ and $l \neq 0$.

2.1. Calculation of energy for $l = 0$ state

For the $l = 0$ state

$$\psi_0 = N J_0(\rho), \tag{10}$$

where $\rho = kr$, J_0 is the 0th order Bessel function, and $N = 1/(\sqrt{\pi} R_0 J_1(\rho_{0,j}))$ is the normalization constant. E_0 is obtained from (6) with $l = 0$, and an appropriate j , as ψ_0 satisfies boundary condition (9a). The first-order correction to the wavefunction, obtained as a solution to (8b) is

$$\psi_1 = \sum_{p=1}^{\infty} (a_p \cos p\theta + \bar{a}_p \sin p\theta) J_p + a_0 J_0 - \frac{\rho E_1}{2E_0} N J_1, \tag{11}$$

where the last term is the particular integral to (8b). Incorporating (11) in (9b) and separately matching the coefficients of the cosine and the sine terms we have

$$a_p = -\rho_{0,j} N C_p^{(1)} \frac{J_0'(\rho_{0,j})}{J_p(\rho_{0,j})}, \tag{12a}$$

$$\bar{a}_p = -\rho_{0,j} N S_p^{(1)} \frac{J_0'(\rho_{0,j})}{J_p(\rho_{0,j})}, \tag{12b}$$

$$E_1 = 0. \tag{12c}$$

The remaining constant a_0 can be found out by normalizing the corrected wavefunction over the enclosed area. However, that is not required right now for our purpose. Equation (12c) implies that there cannot be any correction to the energy in the first order. So any possible correction to the energy can only come from the second or higher orders.

In a similar fashion the second correction to the wavefunction as a solution to (8c) with $E_1 = 0$ is found out to be

$$\psi_2 = \sum_{p=1}^{\infty} (b_p \cos p\theta + \bar{b}_p \sin p\theta) J_p + b_0 J_0 - \frac{\rho E_2}{2E_0} N J_1, \quad (13)$$

which, when introduced in (9c), now yields

$$E_2 = E_0 \left[\sum_{k=1}^{\infty} (C_k^{(1)2} + S_k^{(1)2}) \left[\frac{1}{2} + \rho_{0,j} \frac{J'_k(\rho_{0,j})}{J_k(\rho_{0,j})} \right] - 2C_0^{(2)} \right], \quad (14a)$$

$$b_p = -\rho_{0,j} J'_0(\rho_{0,j}) [N C_p^{(2)} + a_0 C_p^{(1)}] + \frac{\rho_{0,j} J'_0(\rho_{0,j}) N}{2} \sum_{k=1}^{\infty} [C_{p+k}^{(1)} C_k^{(1)} + S_{p+k}^{(1)} S_k^{(1)} + C_{|p-k|}^{(1)} C_k^{(1)} - S_{p-k}^{(1)} S_k^{(1)} + S_{k-p}^{(1)} S_k^{(1)}] \left(\frac{1}{2} + \rho_{0,j} \frac{J'_k(\rho_{0,j})}{J_k(\rho_{0,j})} \right), \quad (14b)$$

$$\bar{b}_p = -\rho_{0,j} J'_0(\rho_{0,j}) [N S_p^{(2)} + a_0 S_p^{(1)}] + \frac{\rho_{0,j} J'_0(\rho_{0,j}) N}{2} \sum_{k=1}^{\infty} [S_{p+k}^{(1)} C_k^{(1)} - C_{p+k}^{(1)} S_k^{(1)} + C_{|p-k|}^{(1)} S_k^{(1)} - S_{k-p}^{(1)} C_k^{(1)} + S_{p-k}^{(1)} C_k^{(1)}] \left(\frac{1}{2} + \rho_{0,j} \frac{J'_k(\rho_{0,j})}{J_k(\rho_{0,j})} \right). \quad (14c)$$

As before, the remaining constant b_0 can be determined by normalizing the wavefunction up to the order of λ^2 .

2.2. Calculation of energy for $l \neq 0$ state

The $l \neq 0$ states come in two varieties

$$\psi_0 = N_l J_l(\rho) \begin{pmatrix} \cos l\theta \\ \sin l\theta \end{pmatrix}, \quad (15)$$

where $N_l = \sqrt{2}/(\sqrt{\pi} R_0 J'_l(\rho_{l,j}))$. E_0 is given by (6). For simplicity, we assume that $S_n^{(\sigma)} = 0$ for all σ . We shall first work with

$$\psi_0 = N_l J_l(\rho) \cos l\theta. \quad (16)$$

The result for the other case will be similar. The first correction to the wavefunction obtained as a solution to (8b) is

$$\psi_1 = \sum_{p=0, p \neq l}^{\infty} a_p J_p \cos p\theta + \left(a_l J_l - \frac{E_1 \rho}{E_0} \frac{N_l J_{l+1}}{2} \right) \cos l\theta. \quad (17)$$

Following a similar procedure as that for the ground state we now have

$$E_1 = -C_{2l}^{(1)} E_0, \quad (18a)$$

$$a_p = -\frac{\rho_{l,j}}{2} N_l \frac{J'_l(\rho_{l,j})}{J_p(\rho_{l,j})} (C_{p+l}^{(1)} + C_{|p-l|}^{(1)}), \quad \text{for } p \neq 0, l, \quad (18b)$$

$$a_0 = -\frac{\rho_{l,j}}{2} N_l \frac{J'_l(\rho_{l,j})}{J_0(\rho_{l,j})} C_l^{(1)}. \quad (18c)$$

a_l can be obtained from the normalization condition. The second-order corrections yield

$$\psi_2 = \sum_{m=0}^{\infty} \left[b_m J_m - \frac{\rho E_1}{2E_0} a_m J_{m+1} \right] \cos m\theta + \left[C_{2l}^{(1)2} \frac{\rho}{4} J_{l+2} - \frac{E_2}{E_0} J_{l+1} \right] N_l \frac{\rho}{2} \cos l\theta, \quad (19a)$$

$$\begin{aligned} \frac{E_2}{E_0} = & \frac{C_{2l}^{(1)2}}{2} + \frac{1}{4} \sum_{n=1}^{\infty} C_n^{(1)} (2C_n^{(1)} + C_{2l+n}^{(1)} + C_{|2l-n|}^{(1)}) - 2C_0^{(2)} - C_{2l}^{(2)} \\ & + \sum_{\substack{n=1 \\ n \neq l}}^{\infty} (C_{n+l}^{(1)} + C_{|n-l|}^{(1)})^2 \frac{\rho_{l,j} J'_n(\rho_{l,j})}{2J_n(\rho_{l,j})} + C_l^{(1)2} \frac{\rho_{l,j} J'_0(\rho_{l,j})}{J_0(\rho_{l,j})}. \end{aligned} \quad (19b)$$

The constants b_m can also be determined as in the case of the ground state. However, they are not needed now but their need would arise when one wished to evaluate the third-order correction for energy. For the case

$$\psi_0 = N_l J_l(\rho) \sin l\theta, \quad (20)$$

similar calculations result in

$$\psi_1 = \sum_{p=1, p \neq l}^{\infty} \bar{a}_p J_p \sin p\theta + \left(\bar{a}_l J_l - \frac{E_1}{E_0} \frac{\rho}{2} N_l J_{l+1} \right) \sin l\theta, \quad (21a)$$

$$E_1 = C_{2l}^{(1)} E_0, \quad (21b)$$

$$\bar{a}_p = \frac{\rho_{l,j}}{2} N_l \frac{J'_l(\rho_{l,j})}{J_p(\rho_{l,j})} (C_{p+l}^{(1)} - C_{|p-l|}^{(1)}), \quad \text{for } p \neq l, \quad (21c)$$

$$\psi_2 = \sum_{m=1}^{\infty} \left[\bar{b}_m J_m - \frac{\rho E_1}{2E_0} \bar{a}_m J_{m+1} \right] \sin m\theta + \left[C_{2l}^{(1)2} \frac{\rho}{4} J_{l+2} - \frac{E_2}{E_0} J_{l+1} \right] N_l \frac{\rho}{2} \sin l\theta, \quad (21d)$$

and

$$\begin{aligned} \frac{E_2}{E_0} = & \frac{C_{2l}^{(1)2}}{2} + \frac{1}{4} \sum_{n=1}^{\infty} C_n^{(1)} (2C_n^{(1)} - C_{2l+n}^{(1)} - C_{|2l-n|}^{(1)}) - 2C_0^{(2)} + C_{2l}^{(2)} \\ & + \sum_{n=1, n \neq l}^{\infty} (C_{n+l}^{(1)} - C_{|n-l|}^{(1)})^2 \frac{\rho_{l,j} J'_n(\rho_{l,j})}{2J_n(\rho_{l,j})}. \end{aligned} \quad (21e)$$

We do not give the expressions for \bar{b}_m , as they are not needed now.

3. Application to simple cases

The general formalism having been outlined above, we now estimate the energy levels of a supercircle and an ellipse where direct comparison with the numerical results can be made. Numerical results were calculated using the finite difference method. Both square and triangular grids were used separately for the numerical simulation. The results agree quite well for both types of grids.

3.1. A particle enclosed in a supercircular enclosure

The Piet Hein superellipse [16] is a special case of Lamé curves described by

$$\frac{|x|^t}{a^t} + \frac{|y|^t}{b^t} = 1, \quad (22)$$

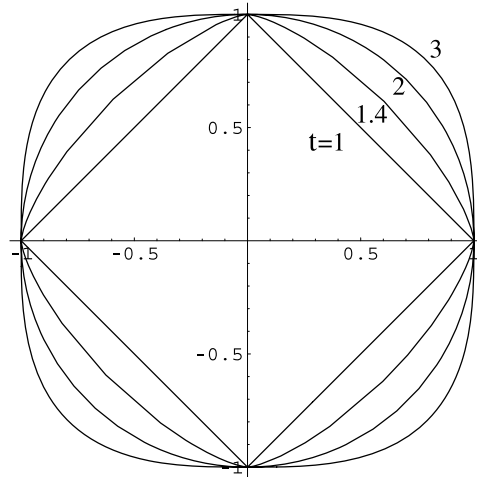


Figure 1. Shape of the supercircle for different values of t .

with $t > 1$. a and b are positive real numbers. They are also known as Lamé curves or Lamé ovals [17]. Superellipses can be parametrically described as

$$x = a \cos^{2/t} \phi \quad \text{and} \quad y = b \sin^{2/t} \phi. \quad (23)$$

Different values of t would give us closed curves of different shapes. For $t > 1$ we consider only the real positive values of $\cos^{2/t} \phi$ and $\sin^{2/t} \phi$ for $0 \leq \phi \leq \frac{\pi}{2}$ and use the symmetry of the figure to continue to the other quadrants. We are interested in the case $a = b$, which corresponds to a supercircle. In polar coordinates the equation for the supercircle is

$$r = \frac{a}{(\cos^t \theta + \sin^t \theta)^{1/t}}, \quad (24)$$

and the radius of the equal area circle is

$$R_0 = a \sqrt{\frac{2}{t\pi}} \frac{[\Gamma(\frac{1}{t})]}{\sqrt{[\Gamma(\frac{2}{t})]}}. \quad (25)$$

The shapes of supercircles for different values of t are shown in figure 1. $t = 2$ describes a circle of unit radius. In this case we have a natural deformation parameter, $\lambda = 2 - t$. Now $r(\theta)$ given by (24) can be Fourier expanded and after some calculation one arrives at the following:

$$r(\theta) = R_0 \left[1 + \lambda \sum_{n=1}^{\infty} C_{4n}^{(1)} \cos 4n\theta + \lambda^2 \sum_{n=0}^{\infty} C_{4n}^{(2)} \cos 4n\theta + O(\lambda^3) \dots \right], \quad (26)$$

where the Fourier coefficients are found to be

$$\begin{aligned} C_{4n}^{(1)} &= -\frac{1}{4n(4n^2 - 1)}, \\ C_0^{(2)} &= -\frac{1}{4} \sum_{n=1}^{\infty} \left[\frac{1}{4n(4n^2 - 1)} \right]^2 \\ &= \frac{1}{16} \left(1 - \frac{5\pi^2}{48} \right) = -0.0017552, \end{aligned}$$

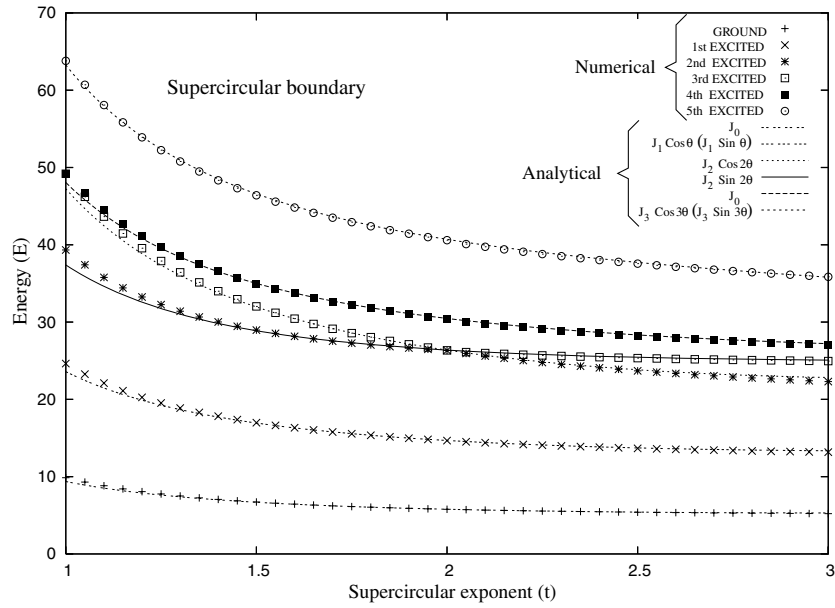


Figure 2. Comparison of the energy eigenvalues obtained numerically and analytically for a particle enclosed in a supercircular boundary shown for the first six states (in units of $\frac{\hbar^2}{2mR_0^2}$). Deformation parameter $\lambda = 2 - t$.

using (5c) and

$$C_4^{(2)} = \frac{1}{32} \left(\frac{3\pi^2}{8} - \frac{23}{9} \right) = 0.0357983.$$

Using these Fourier coefficients, the first six energy levels are calculated for the supercircular boundary in the range $-1 \leq \lambda \leq 1$, and compared with the numerically obtained values. This is shown in figure 2. The numerical results are shown by discrete points and the analytical ones by the continuous lines. The fact that even for such a wide range of λ the analytical results are in fairly good agreement with those obtained numerically does indeed justify the validity of our formalism. We see that for $|\lambda|$ as large as 1 the deviations of analytical values from the numerical ones are within 10%. Furthermore, it is to be noted that the energy level corresponding to the unperturbed wavefunction $\psi_0 = N_2 J_2 \cos 2\theta$ is strongly affected compared to the others and crosses over to its counterpart $\psi_0 = N_2 J_2 \sin 2\theta$ at $\lambda = 0$. This crossing of energy levels is solely induced by the variation in the shape of the boundary of the potential well.

3.2. A particle enclosed in an elliptical enclosure

The determination of the energy eigenvalues of a particle enclosed in 2D with an elliptical boundary has been investigated extensively. In this case the separation of variables is possible in an elliptical coordinate system and the problem is exactly solvable in principle. The problem reduces to solving the Mathieu differential equation for each of the separated coordinates. Extracting out the eigenvalues and the eigenfunctions from the above is a difficult task and often one relies on numerical estimation. So far most of the efforts have been directed at the numerical estimation of the eigenvalues [18–20]. Recently, an analytical method has been

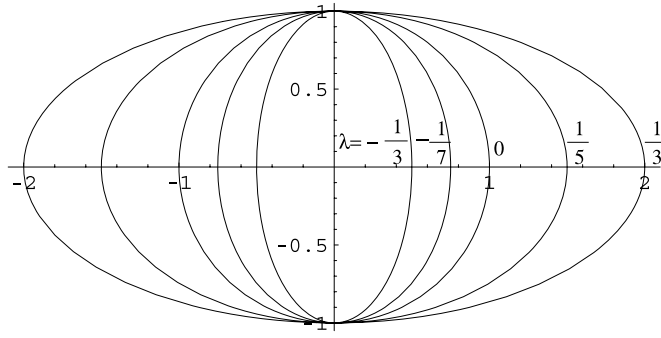


Figure 3. Shape of the ellipse for different values of $\lambda = \frac{a-b}{a+b}$.

suggested by Wu and Shivakumar [21]. Here we propose a simpler approach to the problem by our perturbative method. The equation for an ellipse with semi-axes a and b , in polar coordinates is

$$r(\theta) = \frac{b}{\sqrt{1 - (1 - \frac{b^2}{a^2}) \cos^2 \theta}}. \tag{27}$$

Defining the deformation parameter

$$\lambda = \frac{a - b}{a + b}, \tag{28}$$

we show the shapes of the ellipses for different values of λ in figure 3. Again $\lambda = 0$ describes a circle with unit radius. Now, $r(\theta)$ in (27) can be recast as

$$r = R_0 \left[1 + \lambda \cos 2\theta - \frac{1}{4}\lambda^2 + \frac{3}{4}\lambda^2 \cos 4\theta + O(\lambda^3) + \dots \right], \tag{29}$$

with $R_0 = \sqrt{ab}$. Comparing with our general Fourier series of (4), we observe that

$$C_2^{(1)} = 1, \quad C_0^{(2)} = -\frac{1}{4}, \quad \text{and} \quad C_4^{(2)} = \frac{3}{4}.$$

Using (12c), (14a), (18a), (19b), (21b), (21e) we find

$$E_1 \begin{pmatrix} J_l \cos l\theta \\ J_l \sin l\theta \end{pmatrix} = \begin{pmatrix} - \\ + \end{pmatrix} E_0 \delta_{l1}, \tag{30a}$$

and

$$E_2 \begin{pmatrix} J_l \cos l\theta \\ J_l \sin l\theta \end{pmatrix} = E_0 \sum_{p, |p-l|=2} \left(\frac{\delta_{l1}}{2} + 1 + \frac{\rho_{l,j} J'_p(\rho_{l,j})}{2J_p(\rho_{l,j})} \right) + \begin{pmatrix} -\frac{3}{4} + \frac{\rho_{l,j} J'_0(\rho_{l,j})}{J_0(\rho_{l,j})} \\ \frac{3}{4} \end{pmatrix} E_0 \delta_{2l}, \tag{30b}$$

where δ_{ij} is the Kronecker delta. The results for the elliptical boundary are shown in figure 4. From figure 4 it is seen that as in the case of the supercircle here also the $J_1 \cos \theta$ state is strongly affected by the boundary perturbation and crosses over to its counterpart $J_1 \sin \theta$ at $\lambda = 0$. However, quite interestingly, the J_2 states do not cross but are rather repelled by each other. They touch each other tangentially at $\lambda = 0$. While for one of these states, $J_2 \sin 2\theta$, the analytical method works quite well, it has a restricted validity for the other one,

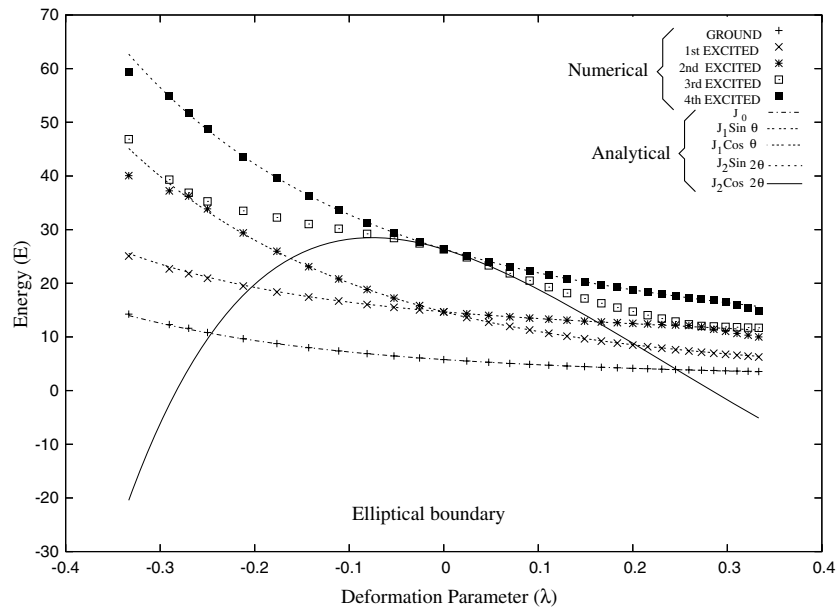


Figure 4. Comparison of the energy eigenvalues obtained numerically and analytically for a particle confined in an elliptical boundary shown for the first five states (in units of $\frac{\hbar^2}{2mR_0^2}$). Deformation parameter $\lambda = \frac{a-b}{a+b}$.

namely, $J_2 \cos 2\theta$. In fact, we compared the energy levels for the first ten states and found the agreement between the analytical and the numerical results to be quite satisfactory except when the levels repel each other. This phenomenon of level repulsion also goes by the name of ‘loci veering’ in the literature. In the case of level repulsion the validity of the perturbation theory for the $J_l \cos l\theta$ states is restricted to a small range in λ (e.g. $|\lambda| \leq 0.08$). This is in sharp contrast to the case where there is no repulsion in which case the agreement with the perturbation theory persists over a wide range.

4. Conclusion

One of the principle virtues of the method proposed is its generality. With slight modification, the formalism can readily be adopted to study the shape dependence of the eigenvalues of a vibrating membrane with Dirichlet conditions on an irregular boundary. The approach can also be useful in studying the modes of propagation of electromagnetic waves in a waveguide with an irregular cross-section. In fact, recently, Dubertrand *et al* [22] have employed a similar scheme for the propagation of electromagnetic waves in open dielectric systems. Another potential area where this formalism might be useful is in the study of quantum dots. This field has been an area of vigorous research for the past few years. 2D quantum dots are generally taken to have a circular symmetry. However, in practice such a symmetry cannot be strictly ensured. There are bound to be small deviations from exact circular symmetry. Hence, probes have been constructed to investigate the shape of the dots [23–25]. As shown in this paper the energy eigenvalues of a particle confined in 2D in an infinitely deep potential well will essentially depend upon the shape of the confining region. Hence a study of the shape dependence of the energy levels might prove to be useful in shedding light upon the actual

shapes of the dots. Another significant aspect of our formalism is the use of the general Fourier series to express the deviation of the boundary from a circular one which allows us to treat any sort of boundary within the limit of small perturbation for which our formalism is valid. Even boundaries with sharp singularities can be treated in our formalism quite efficiently. For example, the square which is a special case of a supercircle with $t = 1$ can be treated quite efficiently by our method. This is borne out by the accuracy of the results obtained by using our formalism in the case of the supercircle for $t = 1$ ($\lambda = 1$), which corresponds to a square (figures 1 and 2). In fact, to find out the energy and the wavefunction corrections all one needs is to find the Fourier coefficients for the closed curve and substitute them in the relevant expressions. Further, the corrections to the energy eigenvalue and the eigenfunctions are found out exactly in a closed form at each order of perturbation without any major approximations which is indeed remarkable. The case of the supercircular boundary shows that even for quite large perturbations the method yields satisfactory results. The accuracy of the method can still be improved by including higher order corrections. In fact, we have also found the third-order corrections, although the results are not included here. In contrast, the case of the elliptical boundary points out to the failure of the perturbation theory whenever the energy levels exhibit repulsion. This provides potential topics for future investigations. Another point which we want to emphasize here is that the success (and also the efficiency) of the formalism depends to a large extent upon the judicious choice of the deformation parameter λ . For the case of the ellipse we defined λ to be equal to $\frac{a-b}{a+b}$ whereas the eccentricity ϵ would seem to be a more appropriate candidate for λ . For the elliptical boundary we have considered deformations up to the extent where $a : b = 2 : 1$ for which $\lambda = 0.333$. Had we formulated the problem in terms of the eccentricity the same deformation would have led to the value of $\epsilon = 0.866$. It can also be shown that in that case the deformation parameter would actually be ϵ^2 , so that for the same deformation we would have $\epsilon^2 = 0.75$ which is obviously much larger than the parameter which we have actually used here. Such a high value of the deformation parameter goes against the very essence of the perturbative nature of the method. This means that while we have terminated the Fourier series and also the eigenvalues at the second order of smallness when working with $\lambda = \frac{a-b}{a+b}$, for $\lambda = \epsilon^2$ we would have to consider higher order terms to get the same accuracy. Finally, we note that the same formalism can also be adopted by perturbing a square or a rectangular boundary for which the results are exactly known.

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